On the thrust due to an air jet flowing from a wing placed in a wind tunnel

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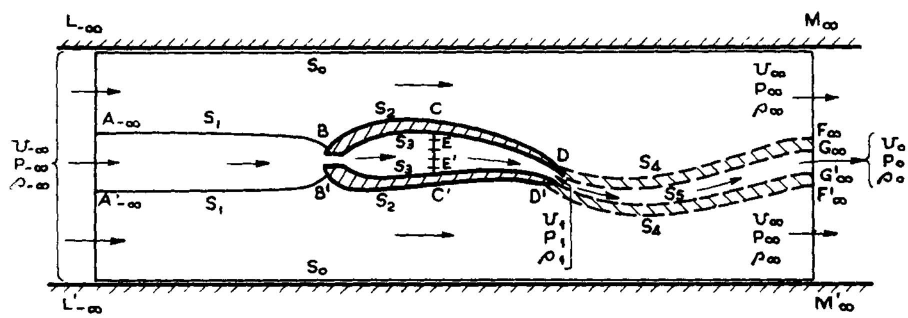
SUMMARY

Consider a wing-jet combination placed in a wind tunnel; the measured thrust on the wing due to a high speed jet emerging from it will require to be corrected to give the infinite stream value. This paper provides a theory of these wind tunnel corrections, and incidently establishes that the ideal thrust is almost independent of the jet exit angle.

1. INTRODUCTION

The figure shows a section through a model of a wing and jet engine combination, the outer and inner surfaces of which are S_2 and S_3 respectively. This model is in a cylindrical wind tunnel, S_0, of arbitrary cross sectional shape. Air is drawn into the wing at BB', the surface S_1 separating the external stream from that drawn into the engine. It is assumed that energy and mass is added to this stream at some section EE' within the wing, converting it into a high speed jet which emerges at some angle to the main



Wing-jet combination in wind tunnel.

stream at the trailing edge DD'. In our idealized mathematical model of the flow this jet is separated from a dead-air region (shown shaded in the figure) by a vortex sheet S_5, and a further vortex sheet S_4 separates the dead-air region from the external flow. The velocity in the dead-air region, which is introduced to represent the blockage effect of the wing drag, is assumed to be zero. Thus the cross sectional area of this region is equal to the displacement area produced by the action of viscosity over the wing surface and internal ducts*.

Let the plane $L_{-\infty}\ L'_{-\infty}$ cut the tunnel at right angles upstream at infinity, and let the curves of intersection of this plane with the surfaces S_0 and S_1 enclose areas A and a respectively. Similarly, let the surfaces S_4 and S_5 have cross sectional areas $(b+h_0)$ and h_0 downstream at infinity, so that b is the displacement area mentioned above.

So far we have assumed in our idealization of the flow that (1) there is no mixing of the jet and main stream fluids, and that (2) the vorticity is confined to vortex sheets. With these assumptions it is natural to assume further that (3) the flow is isentropic in both the jet and the main stream. Thus our model is only a crude first approximation to the real flow, but it does yield what might be termed the "ideal thrust", and a comparison of this with the actual thrust achieved will give a valuable estimate of the loss of jet efficiency due to viscous and turbulent action.

Values of the velocity U, pressure p, and density ρ at infinity, (a) upstream, (b) downstream in the external flow, and (c) downstream in the jet, are distinguished by the subscripts $_{-\infty}$, $_{\infty}$ and $_0$ respectively; the subscript $_1$ is used to denote values in the jet at the trailing edge. The boundary conditions are (1) no flow across tunnel and wing surfaces, (2) continuity of pressure across the dead-air region and the vortex sheets S_4 and S_5, (3) continuity of pressure and velocity across the surface S_1, and (4) known sources of mass and energy within the wing. By (2) the pressure down stream at infinity is p_∞ right across the tunnel, i.e. $p_0=p_\infty$.

2. Exact mathematical theory

Let $\mathbf{n}$ be a unit vector along the outward normal to the (external) region Σ_0 bounded by the surfaces S_0, S_1, S_2, S_4 and the planes $L_{-\infty}\ L'_{-\infty}$, $M_\infty\ M'_\infty$, and $\mathbf{n}'$ a similar outward vector for the (internal) region Σ_1 bounded by S_1, S_3, S_4 and the planes $L_{-\infty}L'_{-\infty}$, $M_\infty\ M'_\infty$. Let $\mathbf{i}$ be a unit vector directed downstream along the tunnel axis. The upstream thrust T acting on the wing may then be written

$$T=-\int_{S_2+S_4} p\mathbf{n}.\mathbf{i}\ dS - \int_{S_3+S_5} p\mathbf{n}'.\mathbf{i}\ dS + p_\infty b. \tag{1}$$

*Alternatively (see Allen & Vincenti 1944) this cross sectional area can be chosen so that it gives the same pressure drop downstream at infinity as that caused by the drag in the actual non-isentropic flow downstream of the model. The displacement thickness method, which is equivalent to assuming isentropic flow, is preferable in the author's opinion, since the non-isentropic flow of the wake will not fill the tunnel for a considerable distance downstream of the model (many tunnel widths), whereas the lower pressure drop of the isentropic flow will be almost completely achieved within a downstream distance of less than one tunnel width. The pressure gradient and increased velocity at the model due to the wake blockage will be produced predominantly by this latter pressure drop.

The right-hand side of this equation contains no drag force, the dead-air region representing only the *displacement* effect of the drag. To represent the drag force fully we must also remove momentum from the stream.

If $\mathbf{q}$ is the velocity vector and S is a surface in the fluid with a normal $\mathbf{n}$, the momentum equation can be written

$$\int_S p\mathbf{n}\, dS = -\int_S (\mathbf{n}.\mathbf{q})\rho\mathbf{q}\, dS.$$

We apply this equation to the regions Σ_0 and Σ_1, and take the component in the $\mathbf{i}$-direction to obtain

$$\int_{S_1+S_2+S_4} p\mathbf{n}.\mathbf{i}\, dS - (A-a)p_{-\infty} + (A-b-h_0)p_\infty$$
$$= (A-a)\rho_{-\infty}U_{-\infty}^2 - (A-b-h_0)\rho_\infty U_\infty^2,$$

$$\int_{S_1+S_3+S_5} p\mathbf{n}'.\mathbf{i}\, dS - ap_{-\infty} + h_0 p_\infty = a\rho_{-\infty}U_{-\infty}^2 - h_0\rho_0 U_0^2.$$

On adding these equations and using (1), we find that

$$T = h_0(\rho_0 U_0^2 - \rho_\infty U_\infty^2) - A(p_{-\infty} + \rho_{-\infty}U_{-\infty}^2 - p_\infty - \rho_\infty U_\infty^2) - b\rho_\infty U_\infty^2. \quad (2)$$

In this equation the upstream values and the areas A and b (see later) are known quantities, and the six quantities p_∞, ρ_∞, U_∞, ρ_0, U_0, h_0 have to be calculated. The six equations necessary for this are

$$(A-a)\rho_{-\infty}U_{-\infty} = (A-b-h_0)\rho_\infty U_\infty, \quad (3)$$

$$a\rho_{-\infty}U_{-\infty} + Q' = h_0\rho_0 U_0, \quad (4)$$

$$\frac{\gamma}{\gamma-1}\frac{p_{-\infty}}{\rho_{-\infty}} + \tfrac{1}{2}U_{-\infty}^2 = \frac{\gamma}{\gamma-1}\frac{p_\infty}{\rho_\infty} + \tfrac{1}{2}U_\infty^2, \quad (5)$$

$$\frac{\gamma'}{\gamma'-1}\frac{p_1}{\rho_1} + \tfrac{1}{2}U_1^2 = \frac{\gamma'}{\gamma'-1}\frac{p_\infty}{\rho_0} + \tfrac{1}{2}U_0^2, \quad (6)$$

$$\frac{p_{-\infty}}{p_\infty} = \left(\frac{\rho_{-\infty}}{\rho_\infty}\right)^\gamma, \quad (7)$$

$$\frac{p_1}{p_\infty} \doteq \left(\frac{\rho_1}{\rho_0}\right)^{\gamma'}, \quad (8)$$

in which Q' is the known strength of the source within the wing and the numbers γ and γ' are the ratios of the specific heats in the main stream and jet respectively. Equations (7) and (8) are adiabatic equations of state, while the first and second pairs follow from the conservation of mass and energy, respectively, for each of the regions Σ_0 and Σ_1.

Equations (2) to (8) contain the exact solution of our problem, but for practical purposes the following approximate theory is sufficient.

3. Approximate theory

For wind tunnels of cross sectional dimensions large relative to the model dimensions, we can set

$$U_{\infty} = U_{-\infty}(1+\delta), \tag{9}$$

where δ is a small first order term, and disregard terms of the second order in δ. Of course, in the limit as the tunnel cross section tends to infinity, $\delta \to 0$, and the approximate theory will yield exact results for an infinite stream. From (5), (7) and (9) it is not difficult to calculate the expansions

$$\rho_{\infty} = \rho_{-\infty}\{1 - M^2\delta + \tfrac{1}{2}M^2\delta^2[(2-\gamma)M^2 - 1] + 0(\delta^3)\}, \tag{10}$$

and

$$p_{\infty} = p_{-\infty} - \rho_{-\infty}U^2_{-\infty}\delta(1+\tfrac{1}{2}\beta^2\delta) + 0(\delta^3), \tag{11}$$

where M is the Mach number upstream at infinity, and $\beta = (1-M^2)^{1/2}$. With these approximations, equation (3) gives $\beta^2\delta = (b+h_0-a)/A$, to the first order in small quantities, and

$$A\beta^2\delta = (b+h_0-a)\left\{1 + \frac{1}{A}\left[b+h_0 - \frac{M^2}{2\beta^4}(b+h_0-a)[(2-\gamma)M^2-3]\right]\right\}, \tag{12}$$

to the second order. Equations (10) to (12) enable us to replace (2) by

$$\frac{T}{\tfrac{1}{2}c\rho_{-\infty}U^2_{-\infty}} = \frac{2h_0\rho_0U_0^2}{c\rho_{-\infty}U^2_{-\infty}} - \frac{2a}{c} - \frac{(b+h_0)^2-a^2}{\beta^2Ac} + 0(\delta^2), \tag{13}$$

where c is the planform area of the wing.

The areas a, b, c and h_0 can be eliminated from (13) by introducing the following coefficients:

the thrust coefficient

$$C_T = \frac{T}{\tfrac{1}{2}c\rho_{-\infty}U^2_{-\infty}}; \tag{14}$$

the momentum coefficient of the jet at infinity,

$$C_{J0} = \frac{h_0\rho_0U_0^2}{\tfrac{1}{2}c\rho_{-\infty}U^2_{-\infty}}, \tag{15}$$

and a similar coefficient C_{J1} referring to conditions at the jet exit; the mass coefficient of the air intake,

$$C_Q = \frac{a\rho_{-\infty}U_{-\infty}}{c\rho_{-\infty}U_{-\infty}} = \frac{a}{c}; \tag{16}$$

the mass coefficient of the source within the wing,

$$C_Q' = \frac{Q'}{c\rho_{-\infty}U_{-\infty}}; \tag{17}$$

and the drag coefficient,

$$C_D = \frac{D}{\tfrac{1}{2}c\rho_{-\infty}U^2_{\infty}} = \frac{b\rho_{-\infty}U^2_{-\infty}}{\tfrac{1}{2}c\rho_{-\infty}U^2_{-\infty}} = \frac{2b}{c}, \tag{18}$$

where D is the drag, which is equal to $b\rho_{-\infty}U^2_{-\infty}$, the rate at which momentum is destroyed.

The momentum coefficients C_{J0} and C_{J1} can be related by using the equations (6), (8), and $h_0 \rho_0 U_0 = h_1 \rho_1 U_1$, the last of which expresses the fact that mass is conserved in the jet. Thus we find that $C_{J0} = C_{J1}(U_0/U_1)$; and if the pressure change along the jet is assumed to be small,

$$C_{J0} = C_{J1}\left\{1 - \frac{p_0 - p_1}{\rho_1 U_1^2}\right\},$$

i.e.

$$C_{J0} = C_{J1} - \frac{2h_1}{c}\frac{p_\infty - p_1}{p_{-\infty} U_{-\infty}^2}, \tag{19}$$

from the definition of C_{J1} and the fact that $p_0 = p_\infty$. However, for simplicity we will express our results in terms of C_{J0}. The drag coefficient with the jet operating may differ appreciably from that for the wing alone. The jet will induce high speeds in the external flow near the trailing edge and will probably prevent any flow separation, but, on the other hand an additional boundary layer will emerge from the jet duct. It will not be easy to determine experimentally the appropriate value of C_D for (18), but as a first approximation the value of C_D for the wing alone could be taken.

From equations (4), (15), (16) and (17), it is found that

$$h_0 = 2c(C_Q + C'_Q)^2/\sigma C_{J0}, \tag{20}$$

where $\sigma = \rho_0/\rho_{-\infty}$ should be near enough to unity in practice. Equations (13) to (20) now give the basic result of this paper, namely

$$C_T = C_{J0} - 2C_Q - \frac{c}{4\beta^2 A}\left\{\left[C_D + \frac{4(C_Q + C'_Q)^2}{\sigma C_{J0}}\right]^2 - 4C_Q^2\right\}. \tag{21}$$

In the limit $A \to \infty$, which corresponds to the case of an infinite stream,

$$C_T = C_{J0} - 2C_Q. \tag{22}$$

Equations (21) and (22) apparently show that C_T is independent of the jet exit angle and the geometry of the wing but this conclusion is not really true. The momentum coefficient that must be adopted for practical purposes is the trailing edge value C_{J1}, and if this value is maintained constant, equation (19) shows that C_{J0} will depend on the pressure p_1 at the trailing edge. In turn, p_1 will depend on the wing shape, and especially on the jet exit angle, τ, say. The determination of the precise dependence of p_1 on τ would demand a much more detailed and accurate model of the flow than the one we have taken. However, it seems clear from (19) that C_T will not vary so rapidly with τ as the function $(1 - \cos\tau)$—the law of dependence our engineering intuition at first suggests. The independence of C_T and τ does follow if C_J is assumed to be constant along the jet. This was first noted by Davidson (1955) for the two-dimensional case. The practical importance of (22) is that, by deflecting jets downwards, lift might be obtained without much corresponding loss in forward thrust, but of course the effects of turbulence and viscosity have yet to be considered. Further, the induced drag could well exceed that produced when lift is obtained in the conventional way by wing incidence. High lift at low speeds rather than 'inexpensive' lift would seem to be the real advantage of jet deflection as a lifting device.

Three special cases of (22) are of some interest. (1) When the jet is derived from a source within the wing alone (as in rockets), $a=0$ in (13), and (22) reduces to

$$C_T = C_{J0}.$$

(2) When there is no jet, but air intakes are operating (as in suction on wings for boundary layer control), (22) gives

$$C_T = -2C_Q,$$

so that there is a 'sink-drag' $2C_Q$ on the wing. This is a well-known result. (3) A special case of (1) occurs when the jet has the same total head as the main stream. From (15) and (17), we find that, in this case, $C_{J0} = 2C'_Q$, and so there is a 'source-thrust' given by

$$C_T = 2C'_Q.$$

4. Wind tunnel corrections

Except for the wake blockage term, the theory of wind tunnel interference on the thrust is almost the same as the theory of the interference on the drag of wings. (For a summary of this latter theory, see Howarth 1953, p. 522.)

Let $U^*_{-\infty}$, $\rho^*_{-\infty}$ be the free stream values corresponding to $U_{-\infty}$, $\rho_{-\infty}$, such that the velocity at a given point on the wing is the same in the free stream as in the tunnel stream. Let ϵ_s, ϵ_w be the solid and wake blockage factors respectively, that is

$$U^*_{-\infty} = U_{-\infty}(1+\epsilon_s+\epsilon_w).$$

The factor ϵ_s depends on the tunnel cross section and model volume, and its values for two- and three-dimensional tunnels have been published (Howarth 1953). On the other hand, ϵ_w is independent of the tunnel and model shape, being equal to one half of the value of δ defined in equation (9). (A semi-infinite disturbance like the wake of a wing always produces half of its final downstream effect of the wing itself.) Thus, from (12),

$$U^*_{-\infty} = U_{-\infty}\left(1+\epsilon_s+\frac{b+h_0-a}{2\beta^2 A}\right). \tag{23}$$

Subtracting (21) from (22) we have

$$C^*_T - C_T = (C_{J0}-2C_Q)\left(\frac{C^*_{J0}-2C^*_Q}{C_{J0}-2C_Q}-1\right)+k,$$

where the stars denote free stream values, and k is the last term in (21). Thus, for fixed values of jet momentum and intake mass,

$$C^*_T = C_T + (C_{J0}-2C_Q)\left(\frac{\rho_{-\infty}U^2_{-\infty}}{\rho^*_{-\infty}U^{*2}_{-\infty}}-1\right)+k,$$

which, by an equation for $\rho^*_{-\infty}$ corresponding to (10), and by (23), we can write as

$$C^*_T = C_T - (2-M^2)\left(\epsilon_s+\frac{b+h_0-a}{2\beta^2 A}\right)(C_{J0}-2C_Q)+k.$$

By (16), (18) and (20), it now follows that

$$C_T^* = C_T - (2-M^2)(C_{J0}-2C_Q)\left(\epsilon_s + \frac{c}{4\beta^2 A}\left\{C_D + 4\frac{(C_Q+C_Q')^2}{\sigma C_{J0}} - 2C_Q\right\}\right)$$

$$+\frac{c}{4\beta^2 A}\left\{\left[C_D + 4\frac{(C_Q+C_Q')^2}{\sigma C_{J0}}\right]^2 - 4C_Q^2\right\}, \qquad (24)$$

which shows how the wind tunnel value C_T may be reduced to the free stream value C_T^*.

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